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Families of vicious walkers

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Abstract

We consider a generalization of the vicious walker problem in which N random walkers in \mathbf{R}^d are grouped into p families. Using field-theoretic renormalization group methods we calculate the asymptotic behaviour of the probability that no pairs of walkers from different families have met up to time t . For $d > 2$, this is constant, but for $d < 2$ it decays as a power $t^{-\alpha}$, which we compute to $\mathcal{O}(\varepsilon^2)$ in an expansion in $\varepsilon = 2 - d$. The second-order term depends on the ratios of the diffusivities of the different families. In two dimensions, we find a logarithmic decay $(\ln t)^{-\bar{\alpha}}$ and compute $\bar{\alpha}$ exactly.

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1. Introduction

Consider the following problem: N random walkers set off from the vicinity of the origin, in d -dimensional Euclidean space, at time $t = 0$. They are divided into p different families: the number of walkers in the j th family is n_j , so that $N = \sum_{j=1}^p n_j$. Within a particular family, walkers are indifferent to each other: their paths may cross. However, each family behaves viciously towards all the others: if two walkers from different families meet, both are annihilated. We may ask many different questions about this problem, but a fundamental quantity is the probability $P(\{n_j\}; t)$ that all the walkers have still survived up to time t . Equivalently, we may consider the ensemble of N independent random walks: $P(\{n_j\}; t)$ is the fraction of these in which no walkers of different families have intersected up to time t .

For a discrete time process on a lattice, if $\mathbf{r}_j^{v_j}(t)$ is the position at time t of the v_j th walker of the j th family, then $P(\{n_j\}; t)$ is the expected value of the indicator function

$$\prod_{t'=0}^t \prod_{1 \leq j < k \leq p} \prod_{v_j=1}^{n_j} \prod_{v_k=1}^{n_k} (1 - \delta(\mathbf{r}_j^{v_j}(t'), \mathbf{r}_k^{v_k}(t'))). \quad (1.1)$$

This problem is of interest for several reasons. It generalizes a number of cases:

1. $n_j = 1$ ($1 \leq j \leq p$) corresponds to *vicious walkers*, a term first introduced by Fisher [1]. It has been studied using ε -expansion methods [2, 3] similar to those of the present paper. The survival probability is known exactly for $d = 1$ in the case when all walkers have the same diffusion constants: it decays as a power $t^{-p(p-1)/4}$ [1, 4, 5]. These methods rely essentially on a fermionic description of the problem [1, 6]. Krattenthaler *et al* [4] introduced the method of the Schur functions, and Katori and Tanemura [5] developed this and discussed the relation with the random matrix theory. These methods do not appear to extend to the case when the diffusion constants are different. Results in this case have been reported for $p = 2$ [1].
2. The case $p = 2$, with $n_1 = n$ and $n_2 = 1$, has been studied by Krapivsky and Redner [7–9] as a model of n predators ('lions') hunting a prey ('lamb'). They were able to obtain exact results for the asymptotic behaviour of the survival probability, again in $d = 1$, for the cases $n = 1, 2$ and arbitrary diffusion constants. For general n , the exponent is related to the smallest eigenvalue of a Dirichlet problem in a certain $(n - 1)$ -dimensional compact region and is probably not expressible analytically, but for large n these authors were able to estimate its behaviour. The 'lion–lamb' problem for $d = 1$ is related to a version of the 'ballot problem' in which it is required to know the probability that one candidate in a ballot remains ahead of the n others at all stages in the poll. Exact results are known only for $n \leq 2$ [10].
3. The 'lion–lamb' problem has another interpretation, in terms of *multiscaling*: if we first regard the trajectory ℓ of the lamb as fixed, and if $p_\ell(t)$ is the probability that it has not been met by a *single* lion, then

$$P(n, 1; t) = \langle p_\ell(t)^n \rangle_\ell \quad (1.2)$$

where the average is over all the realizations of ℓ . The fact that $P(n, 1; t)$ decays with t with an exponent which is not simply linear in n is symptomatic of multiscaling in this problem.

4. More generally, we can regard $P(n_1, n_2, \dots, n_p; t)$ as being the average of the n_1 th power of the survival probability of a *single* walker of family 1, in the presence of (n_2, \dots, n_p) walkers of the $(p - 1)$ other families.
5. Our problem has a strong resemblance to that of the intersection probabilities of Brownian paths. In this case, one studies the ensemble of N random walks in d dimensions each of which begin a distance $\mathcal{O}(a)$ from the origin and which arrive on a hypersphere of radius $r = R \gg a$ before they cross $r = a$, *irrespective* of how long this takes. Once again the walkers are divided into families, and in this case one is interested in the probability $\tilde{P}(\{n_j\}, R, a)$ that the *paths* of walkers of different families do not intersect. Thus, instead of (1.1), \tilde{P} is the expected value of

$$\prod_{1 \leq j < k \leq p} \prod_{v_j=1}^{n_j} \prod_{v_k=1}^{n_k} \prod_{t'=0}^{\infty} \prod_{t''=0}^{\infty} (1 - \delta(\mathbf{r}_j^{v_j}(t'), \mathbf{r}_k^{v_k}(t''))) \quad (1.3)$$

and it is supposed to decay as $(R/a)^{-\tilde{\alpha}}$ as $R/a \rightarrow \infty$, where $\tilde{\alpha}$ depends nontrivially on the $\{n_j\}$. This problem is trivial in $d = 1$, and turns out to have an upper critical dimension $d = 4$, below which an ε -expansion is possible [11]. For $d = 2$ an exact formula for $\tilde{\alpha}(\{n_j\})$ has been derived [12, 13], by exploiting the conformal invariance of the problem.

Given these remarks, it seems important to investigate the general case described in the opening paragraph. As far as we know, the fermionic methods used to attack the vicious walker problem for $d = 1$ do not extend to this case. We have, therefore, employed a

renormalization group (RG) method, which yields, for $d < 2$, results for the exponent $\alpha(\{n_j\})$ of the power-law decay of $P(\{n_j\}; t)$ as a power series in $\varepsilon \equiv 2 - d$. By using field-theoretic methods, the calculation is streamlined, and, once the formalism is set up, involves relatively little explicit calculation. We have carried this computation through $\mathcal{O}(\varepsilon^2)$, and for arbitrary diffusion constants of each family. It would be tedious, but not difficult, to carry it further, as the actual Feynman integrals are elementary. We also show that in two dimensions $P(\{n_j\}; t)$ decays as a universal power of $\ln t$.

The paper is organized as follows: in section 2, for completeness, we collect all our results and show how they reduce in the above-mentioned special cases. In section 3 we set up the field-theoretic formulation of the problem, then in the next section carry out the RG analysis. Section 5 contains a summary and further remarks. Several of the detailed calculations are relegated to appendices.

2. Results

Let p be the number of families, n_j be the number of walkers in the j th family and D_j be their diffusivity. Let $P(\{n_j\}; t)$ be the survival probability

$$P(\{n_j\}; t) = E \left[\prod_{t'=0}^t \prod_{1 \leq j < k \leq p} \prod_{v_j=1}^{n_j} \prod_{v_k=1}^{n_k} (1 - \delta(\mathbf{r}_j^{v_j}(t'), \mathbf{r}_k^{v_k}(t'))) \right]. \tag{2.1}$$

2.1. $d > 2$

In this case there is a finite probability that any pair of walkers will never meet. As a result, $P(\{n_j\}; t)$ approaches a non-universal constant value less than 1, with leading power-law corrections of the form $t^{(2-d)/2}$.

2.2. $d < 2$

$$P(\{n_j\}; t) \sim \text{const } t^{-\alpha(\{n_j\})} \quad \text{as } t \rightarrow \infty \tag{2.2}$$

where, with $\varepsilon = 2 - d$,

$$\alpha = \mathcal{F}_1 \varepsilon + \mathcal{F}_2 \varepsilon^2 + \mathcal{O}(\varepsilon^3) \tag{2.3}$$

with

$$\begin{aligned} \mathcal{F}_1 &= \frac{1}{2} \sum_{1 \leq j_1 < j_2 \leq p} n_{j_1} n_{j_2} = \frac{1}{4} (C_1^2 - C_2) \\ \mathcal{F}_2 &= \frac{1}{2} \sum_{1 \leq j_1 < j_2 < j_3 \leq p} n_{j_1} n_{j_2} n_{j_3} \{ \ln R(D_{j_1}, D_{j_2}, D_{j_3}) + \ln R(D_{j_2}, D_{j_3}, D_{j_1}) \\ &\quad + \ln R(D_{j_3}, D_{j_1}, D_{j_2}) \} + \frac{1}{4} \sum_{1 \leq j_1 < j_2 \leq p} n_{j_1} n_{j_2} \{ (n_{j_1} - 1) \ln R(D_{j_1}, D_{j_2}, D_{j_1}) \\ &\quad + (n_{j_2} - 1) \ln R(D_{j_2}, D_{j_1}, D_{j_2}) \} \\ &= \frac{1}{2} \sum_{1 \leq j_1 < j_2 < j_3 \leq p} n_{j_1} n_{j_2} n_{j_3} \ln \left(\frac{(D_{j_1} D_{j_2} + D_{j_1} D_{j_3} + D_{j_2} D_{j_3})^3}{(D_{j_1} + D_{j_2})^2 (D_{j_2} + D_{j_3})^2 (D_{j_3} + D_{j_1})^2} \right) \end{aligned}$$

$$\begin{aligned}
 & + \frac{1}{4} \sum_{1 \leq j_1 < j_2 \leq p} n_{j_1} n_{j_2} \left\{ n_{j_1} \ln \left(\frac{D_{j_1}(D_{j_1} + 2D_{j_2})}{(D_{j_1} + D_{j_2})^2} \right) + n_{j_2} \ln \left(\frac{(2D_{j_1} + D_{j_2})D_{j_2}}{(D_{j_1} + D_{j_2})^2} \right) \right\} \\
 & - \frac{1}{4} \sum_{1 \leq j_1 \leq j_2 < p} n_{j_1} n_{j_2} \ln \left(\frac{D_{j_1} D_{j_2} (D_{j_1} + 2D_{j_2})(2D_{j_1} + D_{j_2})}{(D_{j_1} + D_{j_2})^4} \right)
 \end{aligned} \tag{2.4}$$

where

$$C_k = \sum_{j=1}^p n_j^k \quad k = 1, 2, \dots \tag{2.5}$$

and

$$R(D_j, D_k, D_\ell) = \frac{D_j D_k + D_j D_\ell + D_k D_\ell}{(D_j + D_k)(D_k + D_\ell)}. \tag{2.6}$$

From this may be deduced various special cases.

2.2.1. *Equal diffusion constants.* Assume that $D_j = D$ for all $j = 1, 2, \dots, p$. Then

$$\alpha = \mathcal{F}_1 \varepsilon + \mathcal{F}_2 \ln \frac{3}{4} \varepsilon^2 + \mathcal{O}(\varepsilon^3) \tag{2.7}$$

with

$$\begin{aligned}
 \mathcal{F}_1 &= \frac{1}{2} \sum_{1 \leq j_1 < j_2 \leq p} n_{j_1} n_{j_2} = \frac{1}{4} (C_1^2 - C_2) \\
 \mathcal{F}_2 &= \frac{3}{2} \sum_{1 \leq j_1 < j_2 < j_3 \leq p} n_{j_1} n_{j_2} n_{j_3} + \frac{1}{4} \sum_{1 \leq j_1 < j_2 \leq p} n_{j_1} n_{j_2} (n_{j_1} + n_{j_2}) - \frac{1}{2} \sum_{1 \leq j_1 < j_2 \leq p} n_{j_1} n_{j_2} \\
 &= \frac{1}{4} (C_1^3 - C_1^2 - 2C_1 C_2 + C_2 + C_3).
 \end{aligned} \tag{2.8}$$

Note that these are expressed in terms of symmetric polynomials in the $\{n_j\}$. This in fact holds to all orders in ε .

2.2.2. *Vicious walkers with unequal diffusion constants.* When

$$n_j = \begin{cases} 1 & \text{for } 1 \leq j \leq p \\ 0 & \text{otherwise} \end{cases} \tag{2.9}$$

α should be equal to the *survival exponent* $\psi_{S,p}$ of the vicious walkers. In this case $C_k = p$ for $k = 1, 2, \dots$ and the result (2.3) gives

$$\begin{aligned}
 \psi_{S,p} &= \alpha|_{n_j=1(1 \leq j \leq p), n_k=0(k \geq p+1)} \\
 &= \frac{1}{2} \binom{p}{2} \varepsilon + \frac{1}{2} \sum_{1 \leq j_1 < j_2 < j_3 \leq p} \ln \left(\frac{(D_{j_1} D_{j_2} + D_{j_1} D_{j_3} + D_{j_2} D_{j_3})^3}{(D_{j_1} + D_{j_2})^2 (D_{j_2} + D_{j_3})^2 (D_{j_3} + D_{j_1})^2} \right) \varepsilon^2 \\
 &+ \mathcal{O}(\varepsilon^3).
 \end{aligned} \tag{2.10}$$

2.2.3. *Vicious walkers with equal diffusion constants.* The result (2.7) gives

$$\begin{aligned}
 \psi_{S,p} &= \alpha|_{n_j=1(1 \leq j \leq p), n_k=0(k \geq p+1)} \\
 &= \frac{1}{2} \binom{p}{2} \varepsilon + \frac{3}{2} \binom{p}{3} \ln \frac{3}{4} \varepsilon^2 + \mathcal{O}(\varepsilon^3) \\
 &= \frac{1}{4} p(p-1) \varepsilon + \frac{1}{4} p(p-1)(p-2) \ln \frac{3}{4} \varepsilon^2 + \mathcal{O}(\varepsilon^3).
 \end{aligned} \tag{2.11}$$

This agrees with the result reported as equation (5.2), with equation (3.13) in Mukherji and Bhattacharjee [2] (see also [3]).

It has been proved that [1, 4, 5]

$$\psi_{S,p} = \frac{1}{4}p(p-1) \quad \text{for } d = 1 \quad (\text{i.e. } \varepsilon = 1). \tag{2.12}$$

Note that although this exact result agrees with that from the first-order ε -expansion (2.11) on setting $\varepsilon = 1$, this is probably fortuitous, as, in the case of unequal diffusivities the exact result depends on their ratio, while the first-order term in (2.10) does not.

2.2.4. ‘Lion–lamb’ problem with unequal diffusion constants. The ‘ n lions and one lamb’ problem studied by Krapivsky and Redner [7, 8] is a special case of the present model in which

$$n_j = \begin{cases} n & \text{for } j = 1 \\ 1 & \text{for } j = 2 \\ 0 & \text{otherwise.} \end{cases} \tag{2.13}$$

In this case $C_k = 1 + n^k$ for $k = 1, 2, \dots$ and the result (2.3) gives

$$\begin{aligned} \beta_n &= \alpha|_{n_1=n, n_2=1, n_j=0(j \geq 3)} \\ &= \frac{1}{2}n\varepsilon + \frac{1}{4}n(n-1) \ln \left(\frac{1+2\eta}{(1+\eta)^2} \right) \varepsilon^2 + \mathcal{O}(\varepsilon^3) \end{aligned} \tag{2.14}$$

where $\eta = D_2/D_1$. Redner and Krapivsky [8] reported the exact solution for $n = 2$ in $d = 1$ (i.e. $\varepsilon = 1$),

$$\beta_2^{\text{exact}}(\eta) = \left[2 - \frac{2}{\pi} \cos^{-1} \frac{\eta}{1+\eta} \right]^{-1}. \tag{2.15}$$

It was shown that $\beta_2^{\text{exact}}(\eta)$ is monotonically decreasing in η and

$$\beta_2^{\text{exact}}(0) = 1 \quad \beta_2^{\text{exact}}(1) = \frac{3}{4} \quad \lim_{\eta \rightarrow \infty} \beta_2^{\text{exact}}(\eta) = \frac{1}{2}. \tag{2.16}$$

If we neglect $\mathcal{O}(\varepsilon^3)$ and set $n = 2, \varepsilon = 1$ in (2.14), we have

$$\beta_2^{\text{approx}}(\eta) = 1 + \frac{1}{2} \ln \left(\frac{1+2\eta}{(1+\eta)^2} \right)$$

which is monotonically decreasing in η and

$$\beta_2^{\text{approx}}(0) = 1 \quad \beta_2^{\text{approx}}(1) = 1 + \frac{1}{2} \ln \frac{3}{4} \simeq 0.856 \quad \lim_{\eta \rightarrow \infty} \beta_2^{\text{approx}}(\eta) = -\infty.$$

2.3. Two dimensions

In this case, there is a logarithmic decay with universal exponent:

$$P(\{n_j\}; t) \sim \text{const} (\ln t)^{-\bar{\alpha}} \left(1 + \mathcal{O} \left(\frac{1}{\ln t} \right) \right) \tag{2.17}$$

where

$$\bar{\alpha} = \sum_{1 \leq j_1 < j_2 \leq p} n_{j_1} n_{j_2}. \tag{2.18}$$

Note that this is independent of the D_j (as long as no pair of them both vanish): the dependence shows up only in the prefactor and the non-leading terms.

3. Field-theoretic formulation

In this section, we set up the general problem as a continuum field theory, so that the powerful techniques of the field-theoretic RG may be applied. The general method for formulating such stochastic particle systems as field theories, as originally proposed by Doi [14] and Peliti [15], has been described at length elsewhere [16], and we shall only summarize how this is applied in the case of interest.

Initially, the problem is formulated on a lattice, for example \mathbf{Z}^d , the sites of which are labelled by a vector \mathbf{r} . The microstate of the system at a given time is specified by occupation numbers $\{m_j(\mathbf{r})\}$, which specify that there are $m_j(\mathbf{r})$ walkers of family j at the site \mathbf{r} . Note that we treat walkers of the same family as identical particles: this makes no difference in the problem of interest. With each microstate is associated a vector in a Fock space \mathcal{F} , built by applying raising operators to the vacuum, or empty state, $|0\rangle$:

$$|\{m_j(\mathbf{r})\}\rangle = \prod_{\mathbf{r} \in \mathbf{Z}^d} \prod_{j=1}^p a_j^\dagger(\mathbf{r})^{m_j(\mathbf{r})} |0\rangle \quad (3.1)$$

where $[a_j(\mathbf{r}), a_k^\dagger(\mathbf{r}')] = \delta_{jk} \delta_{\mathbf{r}\mathbf{r}'}$, and $a_j(\mathbf{r})|0\rangle = 0$. Let $p(\{m_j(\mathbf{r})\}; t)$ be the probability of finding the system in this microstate at time t , and define the state $\in \mathcal{F}$

$$|\Psi(t)\rangle \equiv \sum_{\{m_j(\mathbf{r})\}} p(\{m_j(\mathbf{r})\}; t) |\{m_j(\mathbf{r})\}\rangle. \quad (3.2)$$

Then the master equation, which is linear equation describing the time evolution of the probabilities $p(\{m_j(\mathbf{r})\}; t)$, is equivalent to the Schrödinger-like equation

$$d|\Psi(t)\rangle/dt = -\hat{H}|\Psi(t)\rangle \quad (3.3)$$

where $\hat{H} : \mathcal{F} \rightarrow \mathcal{F}$ may be expressed explicitly in terms of the raising and lowering operators. For the case of independent random walks in continuous time,

$$\hat{H} = \hat{H}_0 = \sum_{j=1}^p (D_j/b^2) \sum_{(\mathbf{r}, \mathbf{r}')} (a_j^\dagger(\mathbf{r}) - a_j^\dagger(\mathbf{r}'))(a_j(\mathbf{r}) - a_j(\mathbf{r}')) \quad (3.4)$$

where b is the lattice spacing, and the sum is over nearest neighbour pairs of sites $(\mathbf{r}, \mathbf{r}')$.

The probability of finding the walkers at sites $\mathbf{r}_j^{v_j}$ at time t (where $1 \leq v_j \leq n_j$ with $1 \leq j \leq p$) is then given by

$$\langle 0 | \prod_{j=1}^p \prod_{v_j=1}^{n_j} a_j(\mathbf{r}_j^{v_j}) e^{-t\hat{H}_0} |\Psi(0)\rangle. \quad (3.5)$$

Of course, when this is summed over all the $\mathbf{r}_j^{v_j}$, it gives unity.

Before considering how to implement the non-intersection constraint, let us first discuss the continuum limit and the path integral representation. In this non-interacting case, the continuum limit may be taken rigorously. The raising and lowering operators go over into (distribution-valued) field operators satisfying $[\phi_j(\mathbf{r}), \phi_k^\dagger(\mathbf{r}')] = \delta_{jk} \delta(\mathbf{r} - \mathbf{r}')$, and the generator of time evolution becomes

$$\hat{H}_0 = \int \left[\sum_{j=1}^p D_j (\nabla \phi_j^\dagger) (\nabla \phi_j) \right] d^d r. \quad (3.6)$$

Since the walkers are all supposed to begin in the vicinity of the origin at $t = 0$, that is a finite number of lattice spacings away, in the continuum limit $b \rightarrow 0$

$$|\Psi(0)\rangle = \mathcal{O}^\dagger |0\rangle \equiv \prod_{j=1}^p (\phi_j^\dagger(\mathbf{0}))^{n_j} |0\rangle. \quad (3.7)$$

The path integral representation is derived [16] by breaking the time interval $(0, t)$ into slices of length Δt , so that the time-evolution operator $e^{-t\hat{H}_0}$ is the product of factors $e^{-\Delta t\hat{H}_0} \approx 1 - \Delta t\hat{H}_0$, and inserting a complete set of coherent states at each time slice. This has the effect of replacing the operators $\phi_j(\mathbf{r})$ and $\phi_j^\dagger(\mathbf{r})$ by time-dependent c -number fields $\phi_j(t, \mathbf{r})$ and $\phi_j^*(t, \mathbf{r})$, respectively. After taking the limit $\Delta t \rightarrow 0$, the matrix element (3.5) becomes a functional integral

$$\int \prod_{j=1}^p \mathcal{D}\phi_j^* \mathcal{D}\phi_j \prod_{j=1}^p \prod_{v_j=1}^{n_j} \phi(t, \mathbf{r}_{j,v_j}^v) \mathcal{O}^*(0, \mathbf{0}) e^{-S_0} \quad (3.8)$$

where

$$S_0 = \int \left[\sum_{j=1}^p \phi_j^* \partial_t \phi_j - \sum_{j=1}^p D_j (\nabla \phi_j^*) (\nabla \phi_j) \right] dt d^d r \quad (3.9)$$

and $\mathcal{O}^* = \prod_{j=1}^p \phi_j^{*n_j}$.

3.1. Interactions

We now discuss how to incorporate the constraint that walkers of different families should not meet. Rather than insert the indicator function (1.1) into the path integral, it more convenient to consider a slightly more general problem in which, before taking the limits $b \rightarrow 0$ and $\Delta t \rightarrow 0$, each set of trajectories is weighted by a factor

$$\prod_{t'=0}^t \prod_{\mathbf{r}} \exp \left(- \sum_{1 \leq j_1 < j_2 \leq p} \lambda_{j_1 j_2} (b^d / \Delta t) m_{j_1}(t', \mathbf{r}) m_{j_2}(t', \mathbf{r}) \right) \quad (3.10)$$

where the $\lambda_{j_1 j_2} (b^d / \Delta t) > 0$ are a set of dimensionless parameters (the factors of b and Δt are inserted to make the continuum limit simpler). The case of strict non-intersection corresponds to the limit $\lambda_{j_1 j_2} \rightarrow \infty$. However, we shall show that, for $d \leq 2$, the leading behaviour is independent of the precise value of these parameters (as long as they are all strictly positive) and, moreover, the RG fixed point, at which non-leading corrections to the asymptotic behaviour disappear, corresponds to the limit of infinite $\lambda_{j_1 j_2}$.

In the formal continuum limit, this corresponds to a modification of the action in the path integral

$$S = S_0 + \sum_{1 \leq j_1 < j_2 \leq p} \lambda_{j_1 j_2} \int \phi_{j_1}^* \phi_{j_2}^* \phi_{j_1} \phi_{j_2} dt d^d r. \quad (3.11)$$

3.2. Feynman rules

The Feynman rules for this theory are very simple [16] and are illustrated in figure 1. We denote averages and correlations with respect to the bare action S_0 by the subscript $_0$: averages with respect to the full action S are denoted by $\langle \cdot \rangle$.

- The Fourier–Laplace transform of the bare propagator

$$\begin{aligned} G_j^{(1,1)}(s, \mathbf{k})_0 &= \int_0^\infty dt \int d^d r e^{-st} e^{i\mathbf{k}\cdot\mathbf{r}} \langle \phi_j(t, \mathbf{r}) \phi_j^*(0, \mathbf{0}) \rangle_0 \\ &= (s + D_j k^2)^{-1} \end{aligned} \quad (3.12)$$

is represented by a line directed towards increasing time (conventionally, right-to-left). In the (t, \mathbf{k}) representation the bare propagator is simply $e^{-D_j k^2 t}$,

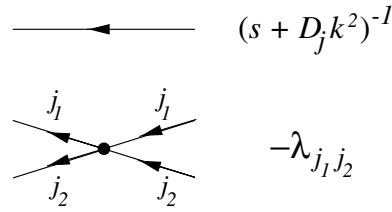


Figure 1. Feynman rules for the interacting theory (3.11). Time always flows towards the left.

- the interaction $(-\lambda_{j_1 j_2})$ is represented by a vertex with one incoming and one outgoing pair of lines each of type j_1 and j_2 , and
- as usual, wave number k and (imaginary) frequency s are conserved at the vertices, and internal loop integrations $\int (ds/2\pi i)$ and $\int (d^d k/(2\pi)^d)$ are carried out.

3.3. Renormalization and operator product expansion

The survival probability is now given by the correlation function

$$\overline{G}_{\mathcal{O}}(t) = \int \prod_{j=1}^p \prod_{v_j=1}^{n_j} \langle \phi_j(t, \mathbf{r}_j^{v_j}) \mathcal{O}^*(0, \mathbf{0}) \rangle \quad (3.13)$$

evaluated with the weight e^{-S} .

However, this does not exist in the formal continuum limit, because the perturbative Feynman diagram expansion of \overline{G} contains ultraviolet (short-distance or short-time) divergences. Physically this is because two walkers, having interacted once, are, in the continuum limit, likely to interact an infinite number of times as $\Delta t \rightarrow 0$. This divergence may be regulated, either by imposing an explicit cut-off $|k| < \Lambda$ in the Feynman integrals, or, more easily, by dimensional regulation. For $d \leq 2$ this field theory is renormalizable: the singular dependence on the regulator may be absorbed into a finite number of parameters. In the case of the theory of interest, this procedure is particularly simple [16]: no renormalization of the field $\phi(t, \mathbf{r})$ nor of the diffusion constants is required, only a simple renormalization of the coupling constants $\lambda_{j j'}$, which can be computed exactly to all orders. The lack of field and diffusion constant renormalization holds mathematically because there are no loop corrections to the propagators. Physically it is because an isolated walker does not interact, even with itself, in the absence of any branching processes. When the coupling constant renormalization is done, all correlation functions of products of ϕ and ϕ^* , at *distinct* spacetime points, have a finite limit as the regulator is removed, when expressed in terms of the renormalized couplings. The fact that the renormalized theory must be defined at some arbitrary scale then leads to RG equations.

However, this procedure is not sufficient to render finite correlation functions involving so-called *composite operators* such as $\mathcal{O}^* = \prod_{j=1}^p \phi_j^*(0, \mathbf{0})^{n_j}$. Physically, this is because if the walkers all begin at exactly the same point, they will all annihilate each other immediately! In order to obtain finite renormalized correlation functions, it is first necessary to point-split the fields:

$$\prod_{j=1}^p \phi_j^*(0, \mathbf{0})^{n_j} \longrightarrow \prod_{j=1}^p \phi_j^*(0, \mathbf{r}_j)^{n_j}. \quad (3.14)$$

(Note that it is not necessary to split the starting points of walkers of the same family, since they do not interact.) Now consider a correlation function of this product with an arbitrary

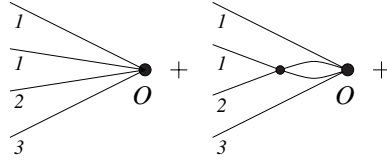


Figure 2. Some of the diagrams contributing to the survival probability \overline{G}_O . The case $n_1 = 2, n_2 = 1, n_3 = 1$ is shown as an illustration. The one-loop diagram shows an interaction between a walker of family 1 with one of family 2 and is proportional to λ_{12} .

product A of fields whose time arguments are all strictly positive:

$$\left\langle A \prod_{j=1}^p \phi_j^*(0, \mathbf{r}_j)^{n_j} \right\rangle. \tag{3.15}$$

In the cut-off theory, we could simply make a Taylor expansion of this in powers of the \mathbf{r}_j . This would have the form

$$\langle A \mathcal{O}^*(0, \mathbf{0}) \rangle + \sum_n C_n(\{\mathbf{r}_j\}) \langle A \mathcal{O}_n^*(0, \mathbf{0}) \rangle \tag{3.16}$$

where the summation is taken over all possible derivatives $\{\mathcal{O}_n^*\}$ of $\prod_j \phi_j^*(0, \mathbf{r}_j)^{n_j}$ with respect to the $\{\mathbf{r}_j\}$. On the basis of dimensional analysis, the first term gives the leading behaviour for $\overline{G}_O(t)$ as $t \rightarrow \infty$, at least at $d = 2$. In the interacting theory, however, each term in (3.16) has to be renormalized separately. As a result

$$\left\langle A \prod_{j=1}^p \phi_j^*(0, \mathbf{r}_j)^{n_j} \right\rangle = Z_O^{-1} \langle A \mathcal{O}_R^* \rangle + \sum_n Z_{O_n}^{-1} C_{n,R}(\{\mathbf{r}_j\}) \langle A \mathcal{O}_{n,R}^* \rangle \tag{3.17}$$

where all the correlation functions are finite as the regulator is removed. Since each of the $\{\mathcal{O}_{n,R}\}$ may acquire a nontrivial anomalous dimension through this procedure, the renormalized functions $C_{n,R}(\{\mathbf{r}_j\})$ have a nontrivial dependence on their arguments. The important feature of (3.17) is that the renormalization constants Z_O, Z_{O_n} are independent of A . This we may write as the *operator product expansion* (OPE)

$$\prod_{j=1}^p \phi_j^*(0, \mathbf{r}_j)^{n_j} = Z_O^{-1} \mathcal{O}_R^*(0, \mathbf{0}) + \sum_n Z_{O_n}^{-1} C_{n,R}(\{\mathbf{r}_j\}) \mathcal{O}_{n,R}^*(0, \mathbf{0}) \tag{3.18}$$

where the OPE functions $C_{n,R}(\{\mathbf{r}_j\})$, etc are in general nontrivial.

Each term in the OPE (3.18), when substituted into \overline{G} , will give rise to nontrivial power-law dependence on t for $d \leq 2$. However, since in the non-interacting theory we know that only the first term is important as $t \rightarrow \infty$, we shall assume that this remains true for sufficiently small ε . Further discussion of this point will be postponed to section 5.

3.4. $d > 2$

In the absence of any interactions, the survival probability $\overline{G}_O(t) = 1$, as can be seen by evaluating the first diagram in the expansion shown in figure 2 with all the wave numbers $\mathbf{q}_j^{v_j}$ set to $\mathbf{0}$. For $d > 2$, the higher-order terms give contributions which correct this constant, as well as terms which are subleading as $t \rightarrow \infty$. For example, a typical one-loop diagram like that in figure 2 is proportional to

$$-\lambda_{j_1 j_2} \int_0^t dt' \int d^d k e^{-(D_{j_1} + D_{j_2}) \mathbf{k}^2 t'} \propto - \int \frac{1 - e^{-(D_{j_1} + D_{j_2}) \mathbf{k}^2 t}}{\mathbf{k}^2} d^d k. \tag{3.19}$$

This integral diverges at large $|\mathbf{k}|$ for $d \geq 2$, a consequence of taking the naive continuum limit. If we impose a cut-off $|\mathbf{k}| < \Lambda = \mathcal{O}(b)$, the leading term behaves as Λ^{d-2} , with a

non-universal coefficient, while the remainder is finite as $\Lambda \rightarrow \infty$ and behaves as $t^{-(d-2)/2}$. The non-universal constant term corresponds to a finite probability that the walkers annihilate at short times, before escaping each other. This behaviour persists at higher orders in the interactions.

For $d \leq 2$, however, each successive term in the bare perturbation grows as a larger and larger positive power of t , and it is necessary to resum the expansion. The RG provides a consistent framework within which to carry this out.

4. Renormalization group analysis

4.1. Coupling constant renormalization

As usual, the renormalized couplings λ_{Rjk} are defined as the values at the normalization point of the irreducible vertex functions $\Gamma_{jk}^{(2,2)}((s'_j, \mathbf{q}'_j), (s'_k, \mathbf{q}'_k); (s_j, \mathbf{q}_j), (s_k, \mathbf{q}_k))$, which are the truncated Laplace–Fourier transforms of $\langle \phi_j(t'_j, \mathbf{r}'_j) \phi_j(t'_k, \mathbf{r}'_k) \phi_j^*(t_j, \mathbf{r}_j) \phi_j^*(t_k, \mathbf{r}_k) \rangle$. (There is no field renormalization in this theory.) It is convenient to choose the normalization point as

$$s'_j = s'_k = s_j = s_k = \sigma \neq 0 \quad \mathbf{q}'_j = \mathbf{q}'_k = \mathbf{q}_j = \mathbf{q}_k = 0. \quad (4.1)$$

This class of theories has the special property that the renormalized coupling constants may be computed to all orders [16]. The calculation is summarized in appendix A. The result is

$$\lambda_{Rjk} = \frac{\lambda_{jk}}{1 + (b_d/\varepsilon)\lambda_{jk}((D_j + D_k)/2)^{-d/2}(2\sigma)^{-\varepsilon/2}} \quad (4.2)$$

where

$$b_d \equiv \frac{2-d}{2^{3d/2}\pi^{d/2}}\Gamma(1-d/2) = 1/(4\pi) + \mathcal{O}(\varepsilon). \quad (4.3)$$

It is convenient to define the dimensionless renormalized couplings as

$$g_{Rjk} \equiv \lambda_{Rjk} \left(\frac{2}{D_j + D_k} \right)^{d/2} (2\sigma)^{-\varepsilon/2}. \quad (4.4)$$

As will be seen, these are the natural expansion parameters for the renormalized perturbation expansion.

4.2. Renormalization of \mathcal{O}^*

As discussed in the previous section, we are interested in computing the asymptotic behaviour at large t of

$$\overline{G}_{\mathcal{O}}(t) = \int \prod_{j=1}^p \prod_{v_j=1}^{n_j} d^d r_j^{v_j} \langle \phi_j(t, \mathbf{r}_j^{v_j}) \mathcal{O}^*(0, \mathbf{0}) \rangle \quad (4.5)$$

in the regularized bare theory, where $\mathcal{O}^* = \prod_{j=1}^p (\phi_j^*)^{n_j}$. However, for the purposes of renormalizing \mathcal{O}^* , it is more convenient to choose the time arguments of the fields ϕ_j to be independent, and to consider the Laplace transform with respect to these times. Thus we define

$$G_{\mathcal{O}}(\{t_j^{v_j}\}) = \int \prod_{j=1}^p \prod_{v_j=1}^{n_j} d^d r_j^{v_j} \langle \phi_j(t_j^{v_j}, \mathbf{r}_j^{v_j}) \mathcal{O}^*(0, \mathbf{0}) \rangle \quad (4.6)$$

and

$$\tilde{G}_{\mathcal{O}}(\{s_j^{v_j}\}) = \int_0^\infty \prod_{j=1}^p \prod_{v_j=1}^{n_j} \{ dt_j^{v_j} e^{-s_j^{v_j} t_j^{v_j}} \} G_{\mathcal{O}}(\{t_j^{v_j}\}). \quad (4.7)$$

From this we may define the irreducible vertex function [17], by truncating the external propagators:

$$\Gamma_{\mathcal{O}}(\{s_j^{v_j}\}; \{\lambda_{jk}\}) = \frac{\tilde{G}_{\mathcal{O}}(\{s_j^{v_j}\})}{\prod_{j=1}^p \prod_{v_j=1}^{n_j} (s_j^{v_j})^{-1}}. \tag{4.8}$$

The renormalized vertex function is

$$\Gamma_{\mathcal{O}R}(\{s_j^{v_j}\}; \{g_{Rjk}\}, \sigma) = Z_{\mathcal{O}}(\{\lambda_{jk}\}, \sigma) \Gamma_{\mathcal{O}}(\{s_j^{v_j}\}; \{\lambda_{jk}\}) \tag{4.9}$$

where $Z_{\mathcal{O}}$ is fixed by the normalization condition

$$\Gamma_{\mathcal{O}R}(\{s_j^{v_j}\}; \{g_{Rjk}\}, \sigma) \Big|_{s_1^1=s_1^2=\dots=s_p^{n_p}=\sigma} = 1. \tag{4.10}$$

In writing (4.9), we have made it clear that that the (un)renormalized vertex function is to be thought of as depending on the (un)renormalized couplings.

Now, although we have used the condition (4.10) on $\Gamma_{\mathcal{O}R}$ to define $Z_{\mathcal{O}}$, the *same* multiplicative renormalization also renders

$$\overline{G}_{\mathcal{O}R}(t; \{\lambda_{Rjk}\}, \sigma) = Z_{\mathcal{O}} \overline{G}_{\mathcal{O}}(t; \{\lambda_{jk}\}) \tag{4.11}$$

finite, for $t > 0$, where $\overline{G}_{\mathcal{O}}$ is defined in (4.5). For this to be true, it is important that the fields $\phi_j(t, \mathbf{r}_j^{v_j})$ are not evaluated at the same point. This would lead to further UV divergences. However, these occur on a set of measure zero in the integration in (4.5) and are harmless.

4.3. Callan–Symanzik equation for $\overline{G}_{\mathcal{O}}$

Define the RG functions

$$\beta_{jk}(g_{Rjk}) = \sigma \left(\frac{\partial g_{Rjk}}{\partial \sigma} \right)_{\{\lambda_{jk}, D_j\}} \tag{4.12}$$

and

$$\gamma_{\mathcal{O}}(\{g_{Rjk}\}) = \left(\sigma \frac{\partial}{\partial \sigma} \ln Z_{\mathcal{O}}(\{\lambda_{jk}\}, \sigma) \right)_{\{\lambda_{jk}\}}. \tag{4.13}$$

The fact that $\sigma(\partial/\partial\sigma)\overline{G}_{\mathcal{O}}|_{\{\lambda_{jk}\}} = 0$ then implies the Callan–Symanzik equation

$$\left(\sigma \frac{\partial}{\partial \sigma} - \gamma_{\mathcal{O}}(\{g_{Rjk}\}) + \sum_{1 \leq j < k \leq p} \beta_{jk}(g_{Rjk}) \frac{\partial}{\partial g_{Rjk}} \right) \overline{G}_{\mathcal{O}R}(t, \{g_{Rjk}\}, \sigma) = 0. \tag{4.14}$$

If the couplings $\{g_{Rjk}\}$ flow towards a nontrivial fixed point $\{g_{Rjk}^*\}$ at which $\beta_{jk}(g_{Rjk}^*) = 0$ (as we shall show happens for $d < 2$), then in estimating the leading asymptotic behaviour as $\sigma \rightarrow \infty$ it is sufficient to replace (4.14) by

$$\left(\sigma \frac{\partial}{\partial \sigma} - \gamma_{\mathcal{O}}^* \right) \overline{G}_{\mathcal{O}R}(t, \sigma) = 0 \tag{4.15}$$

where $\gamma_{\mathcal{O}}^* = \gamma_{\mathcal{O}}(\{g_{Rjk}^*\})$. This has the solution $\overline{G}_{\mathcal{O}R}(t, \sigma) \propto \sigma^{\gamma_{\mathcal{O}}^*}$, as $\sigma \rightarrow \infty$ at fixed t .

However, simple dimensional analysis implies that $\overline{G}_{\mathcal{O}R}(t, \sigma)$ is function of only the combination (σt) . Hence we find that

$$\overline{G}_{\mathcal{O}R}(t, \sigma) \sim \text{const } t^{-\alpha} \tag{4.16}$$

with

$$\alpha = -\gamma_{\mathcal{O}}^*. \tag{4.17}$$

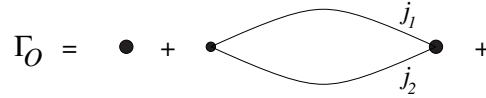


Figure 3. One-loop diagrams for $\Gamma_{\mathcal{O}}$.

4.4. β -functions

From (4.2) and (4.4), we find after some algebra (see appendix A)

$$\beta_{jk}(g_{Rjk}) = -\frac{1}{2}(\varepsilon g_{Rjk} - b_d g_{Rjk}^2). \quad (4.18)$$

Note that this is exact to all orders in g_{Rjk} , and that, for $\varepsilon > 0$, there is an infrared stable fixed point at

$$g_{Rjk}^* = \varepsilon/b_d = 4\pi\varepsilon + \mathcal{O}(\varepsilon^2) \quad (4.19)$$

whose value is independent of the diffusion constants.

4.5. One-loop calculation of $\gamma_{\mathcal{O}}$

Consider the expansion of the vertex function (4.8) as a power series in the coupling constants $\{\lambda_{jk}\}$. To first order, this is given by the sum of diagrams like that in figure 3, explicitly

$$\Gamma_{\mathcal{O}} = 1 - \sum_{1 \leq j_1 < j_2 \leq p} n_{j_1} n_{j_2} \lambda_{j_1 j_2} I_1(s_{j_1}^\mu, s_{j_2}^\nu; D_{j_1}, D_{j_2}) + \mathcal{O}(\{\lambda_{jk}^2\}) \quad (4.20)$$

where the integral $I_1(s_{j_1}^\mu, s_{j_2}^\nu; D_{j_1}, D_{j_2})$ is the same as occurs in the coupling constant renormalization: see appendix A. The combinatorial factor $n_{j_1} n_{j_2}$ counts the number of ways the walkers from family j_1 can interact just once with those of family j_2 . The renormalization constant $Z_{\mathcal{O}}$ is then the inverse of this evaluated at the normalization point $s_j^{v_j} = \sigma$. Thus

$$\ln Z_{\mathcal{O}} = \sum_{1 \leq j_1 < j_2 \leq p} n_{j_1} n_{j_2} \lambda_{j_1 j_2} \frac{b_d}{\varepsilon} \left(\frac{2}{D_{j_1} + D_{j_2}} \right)^{d/2} (2\sigma)^{-\varepsilon/2} + \mathcal{O}(\{\lambda_{jk}^2\}) \quad (4.21)$$

and so, by (4.13)

$$\begin{aligned} \gamma_{\mathcal{O}} &= -\frac{1}{2} \sum_{1 \leq j_1 < j_2 \leq p} n_{j_1} n_{j_2} \lambda_{j_1 j_2} b_d \left(\frac{2}{D_{j_1} + D_{j_2}} \right)^{d/2} (2\sigma)^{-\varepsilon/2} + \mathcal{O}(\{\lambda_{jk}^2\}) \\ &= -\frac{1}{2} \sum_{1 \leq j_1 < j_2 \leq p} n_{j_1} n_{j_2} \times b_d g_{Rj_1 j_2} + \mathcal{O}(\{g_{Rjk}^2\}). \end{aligned} \quad (4.22)$$

Next we set $g_{Rj_1 j_2} = g^*$. By (4.19),

$$\gamma_{\mathcal{O}}^* = -\frac{1}{2} \sum_{1 \leq j_1 < j_2 \leq p} n_{j_1} n_{j_2} \varepsilon + \mathcal{O}(\varepsilon^2). \quad (4.23)$$

Through (4.17), this gives the result (2.3) up to $\mathcal{O}(\varepsilon)$. We remark that to this order the result is independent of the diffusion constants, as long as no pair $D_{j_1} + D_{j_2}$ vanishes. This last is of course a pathological case, since then the two families are immobile and cannot meet.

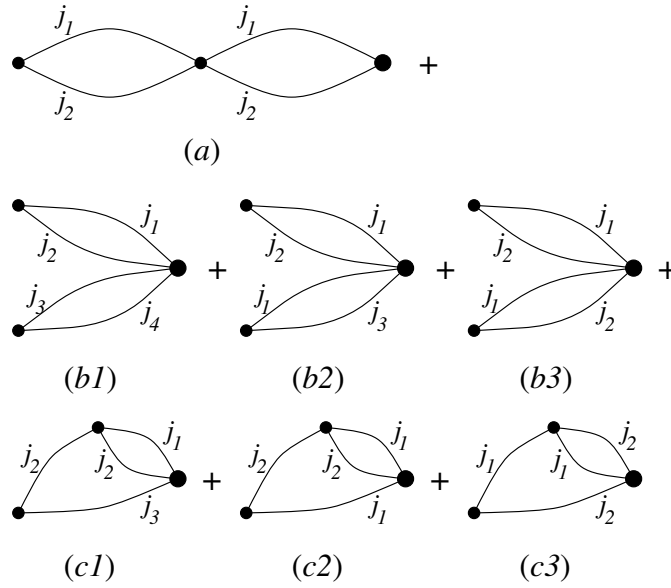


Figure 4. Two-loop contributions to Γ_\emptyset . Each diagram (with possible permutations of the labels) corresponds to a term in (4.26).

4.6. Two-loop calculation

There are three types of diagrams contributing to Γ_\emptyset at order λ^2 . They are illustrated in figures 4(a)–(c). Contributions of types (a) and (b) involve the one-loop integrals I_1 defined above, while those of type (c) involve

$$\begin{aligned}
 & I_2(s_{j_1}^\mu, s_{j_2}^\nu, s_{j_3}^\rho; D_{j_1}, D_{j_2}, D_{j_3}) \\
 &= \iint \frac{(d^d q / (2\pi)^d)(d^d k / (2\pi)^d)}{\{(s_{j_2}^\nu + s_{j_3}^\rho) + (D_{j_2} + D_{j_3})(\mathbf{k} + \mathbf{q})^2\} \{s_{j_1}^\mu + s_{j_2}^\nu + s_{j_3}^\rho + D_{j_1} \mathbf{k}^2 + D_{j_2} \mathbf{q}^2 + D_{j_3} (\mathbf{k} + \mathbf{q})^2\}}.
 \end{aligned}
 \tag{4.24}$$

Define

$$\begin{aligned}
 \hat{I}_1(\sigma; D_{j_1}, D_{j_2}) &= I_1(s_{j_1}^\mu, s_{j_2}^\nu; D_{j_1}, D_{j_2})|_{s_{j_1}^\mu = s_{j_2}^\nu = \sigma} \\
 \hat{I}_2(\sigma; D_{j_1}, D_{j_2}, D_{j_3}) &= I_2(s_{j_1}^\mu, s_{j_2}^\nu, s_{j_3}^\rho; D_{j_1}, D_{j_2}, D_{j_3})|_{s_{j_1}^\mu = s_{j_2}^\nu = s_{j_3}^\rho = \sigma}.
 \end{aligned}
 \tag{4.25}$$

Then

$$\begin{aligned}
 Z_\emptyset^{-1} &= 1 - \sum_{1 \leq j_1 < j_2 \leq p} n_{j_1} n_{j_2} \lambda_{j_1 j_2} \hat{I}_1(\sigma; D_{j_1}, D_{j_2}) + \sum_{1 \leq j_1 < j_2 \leq p} n_{j_1} n_{j_2} (\lambda_{j_1 j_2})^2 (\hat{I}_1(\sigma; D_{j_1}, D_{j_2}))^2 \\
 &+ \sum_{1 \leq j_1 < j_2 < j_3 < j_4 \leq p} n_{j_1} n_{j_2} n_{j_3} n_{j_4} \left\{ \lambda_{j_1 j_2} \lambda_{j_3 j_4} \hat{I}_1(\sigma; D_{j_1}, D_{j_2}) \hat{I}_1(\sigma; D_{j_3}, D_{j_4}) \right. \\
 &+ \lambda_{j_1 j_3} \lambda_{j_2 j_4} \hat{I}_1(\sigma; D_{j_1}, D_{j_3}) \hat{I}_1(\sigma; D_{j_2}, D_{j_4}) \\
 &+ \left. \lambda_{j_1 j_4} \lambda_{j_2 j_3} \hat{I}_1(\sigma; D_{j_1}, D_{j_4}) \hat{I}_1(\sigma; D_{j_2}, D_{j_3}) \right\} \\
 &+ \sum_{1 \leq j_1 < j_2 < j_3 \leq p} \{n_{j_1} (n_{j_1} - 1) n_{j_2} n_{j_3} \lambda_{j_1 j_2} \lambda_{j_1 j_3} \hat{I}_1(\sigma; D_{j_1}, D_{j_2}) \hat{I}_1(\sigma; D_{j_1}, D_{j_3})\}
 \end{aligned}$$

$$\begin{aligned}
& + n_{j_1} n_{j_2} (n_{j_2} - 1) n_{j_3} \lambda_{j_1 j_2} \lambda_{j_2 j_3} \hat{I}_1(\sigma; D_{j_1}, D_{j_2}) \hat{I}_1(\sigma; D_{j_2}, D_{j_3}) \\
& + n_{j_1} n_{j_2} n_{j_3} (n_{j_3} - 1) \lambda_{j_1 j_3} \lambda_{j_2 j_3} \hat{I}_1(\sigma; D_{j_1}, D_{j_3}) \hat{I}_1(\sigma; D_{j_2}, D_{j_3}) \} \\
& + \frac{1}{2} \sum_{1 \leq j_1 < j_2 \leq p} n_{j_1} (n_{j_1} - 1) n_{j_2} (n_{j_2} - 1) (\hat{I}_1(\sigma; D_{j_1}, D_{j_2}))^2 \\
& + \sum_{1 \leq j_1 < j_2 < j_3 \leq p} n_{j_1} n_{j_2} n_{j_3} \{ \lambda_{j_1 j_2} \lambda_{j_2 j_3} \hat{I}_2(\sigma; D_{j_1}, D_{j_2}, D_{j_3}) \\
& + \lambda_{j_1 j_2} \lambda_{j_1 j_3} \hat{I}_2(\sigma; D_{j_2}, D_{j_1}, D_{j_3}) \\
& + \lambda_{j_1 j_3} \lambda_{j_2 j_3} \hat{I}_2(\sigma; D_{j_1}, D_{j_3}, D_{j_2}) + \lambda_{j_1 j_3} \lambda_{j_1 j_2} \hat{I}_2(\sigma; D_{j_3}, D_{j_1}, D_{j_2}) \\
& + \lambda_{j_2 j_3} \lambda_{j_1 j_3} \hat{I}_2(\sigma; D_{j_2}, D_{j_3}, D_{j_1}) + \lambda_{j_2 j_3} \lambda_{j_1 j_2} \hat{I}_2(\sigma; D_{j_3}, D_{j_2}, D_{j_1}) \} \\
& + \sum_{1 \leq j_1 < j_2 \leq p} n_{j_1} (n_{j_1} - 1) n_{j_2} (\lambda_{j_1 j_2})^2 \hat{I}_2(\sigma; D_{j_1}, D_{j_2}, D_{j_1}) \\
& + \sum_{1 \leq j_1 < j_2 \leq p} n_{j_1} n_{j_2} (n_{j_2} - 1) (\lambda_{j_1 j_2})^2 \hat{I}_2(\sigma; D_{j_2}, D_{j_1}, D_{j_2}) + \mathcal{O}(\{\lambda_{jk}^3\}). \quad (4.26)
\end{aligned}$$

Each term in this sum corresponds to a diagram of class (a) to (c3) in figure 4. The combinatorial factors, polynomials in the n_j , count the number of ways different walkers from a given family can contribute to each of these processes. (It is simplest to check these factors for small values of p and n_j .) From appendix A

$$\hat{I}_1(\sigma; D_j, D_k) = \frac{b_d}{\varepsilon} \left(\frac{2}{D_j + D_k} \right)^{d/2} (2\sigma)^{-\varepsilon/2}. \quad (4.27)$$

Moreover, as shown in appendix B

$$\begin{aligned}
\hat{I}_2(\sigma; D_j, D_k, D_\ell) & = \left(\frac{2}{D_j + D_k} \right)^{d/2} \left(\frac{2}{D_k + D_\ell} \right)^{d/2} \\
& \times \left[\frac{1}{2\varepsilon^2} - \frac{1}{4\varepsilon} \ln R(D_j, D_k, D_\ell) + \mathcal{O}(1) \right] b_d^2 (2\sigma)^{-\varepsilon} \quad (4.28)
\end{aligned}$$

where b_d is given by (4.3) and $R(D_j, D_k, D_\ell)$ is given by (2.6).

Next we compute $\ln Z_{\mathcal{O}}$ through $\mathcal{O}(\{\lambda_{jk}^2\})$, perform the differentiation $\sigma \partial / \partial \sigma$ at fixed bare couplings λ_{jk} , then re-express the result as a series in the g_{Rjk} , to the same order. The result has the form

$$\gamma_{\mathcal{O}}(\{g_{Rjk}\}) = B_1 + \frac{1}{\varepsilon} B_2 b_d^2 + B_3 + \mathcal{O}(\{g_{Rjk}^3\}) \quad (4.29)$$

with

$$\begin{aligned}
B_1 & = -\frac{1}{2} \sum_{1 \leq j_1 < j_2 \leq p} n_{j_1} n_{j_2} b_d g_{Rj_1 j_2} \\
B_2 & = -\frac{1}{2} \sum_{1 \leq j_1 < j_2 \leq p} n_{j_1} n_{j_2} (g_{Rj_1 j_2})^2 + \sum_{1 \leq j_1 < j_2 \leq p} n_{j_1} n_{j_2} (g_{Rj_1 j_2})^2 \\
& + \sum_{1 \leq j_1 < j_2 < j_3 < j_4 \leq p} n_{j_1} n_{j_2} n_{j_3} n_{j_4} (g_{Rj_1 j_2} g_{Rj_3 j_4} + g_{Rj_1 j_3} g_{Rj_2 j_4} + g_{Rj_1 j_4} g_{Rj_2 j_3}) \\
& + \sum_{1 \leq j_1 < j_2 < j_3 \leq p} n_{j_1} n_{j_2} n_{j_3} \{ (n_{j_1} - 1) g_{Rj_1 j_2} g_{Rj_1 j_3} + (n_{j_2} - 1) g_{Rj_1 j_2} g_{Rj_2 j_3} \}
\end{aligned}$$

$$\begin{aligned}
 & + (n_{j_3} - 1)g_{R_{j_1 j_3}}g_{R_{j_2 j_3}} \Big\} + \frac{1}{2} \sum_{1 \leq j_1 < j_2 \leq p} n_{j_1} (n_{j_1} - 1) n_{j_2} (n_{j_2} - 1) (g_{R_{j_1 j_2}})^2 \\
 & + \sum_{1 \leq j_1 < j_2 < j_3 \leq p} n_{j_1} n_{j_2} n_{j_3} (g_{R_{j_1 j_2}}g_{R_{j_2 j_3}} + g_{R_{j_2 j_3}}g_{R_{j_1 j_3}} + g_{R_{j_1 j_3}}g_{R_{j_1 j_2}}) \\
 & + \frac{1}{2} \sum_{1 \leq j_1 < j_2 \leq p} n_{j_1} n_{j_2} (n_{j_1} + n_{j_2} - 2) (g_{R_{j_1 j_2}})^2 - \frac{1}{2} \left(\sum_{1 \leq j_1 < j_2 \leq p} n_{j_1} n_{j_2} g_{R_{j_1 j_2}} \right)^2 \\
 B_3 = & -\frac{1}{2} \sum_{1 \leq j_1 < j_2 < j_3 \leq p} n_{j_1} n_{j_2} n_{j_3} \{g_{R_{j_1 j_2}}g_{R_{j_2 j_3}} \ln R(D_{j_1}, D_{j_2}, D_{j_3}) \\
 & + g_{R_{j_2 j_3}}g_{R_{j_1 j_3}} \ln R(D_{j_2}, D_{j_3}, D_{j_1}) + g_{R_{j_1 j_3}}g_{R_{j_1 j_2}} \ln R(D_{j_3}, D_{j_1}, D_{j_2})\} b_d^2 \\
 & - \frac{1}{4} \sum_{1 \leq j_1 < j_2 \leq p} n_{j_1} n_{j_2} \{(n_{j_1} - 1) \ln R(D_{j_1}, D_{j_2}, D_{j_1}) \\
 & + (n_{j_2} - 1) \ln R(D_{j_2}, D_{j_1}, D_{j_2})\} b_d^2 (g_{R_{j_1 j_2}})^2.
 \end{aligned} \tag{4.30}$$

An important check of this calculation is that the double poles in ε in the two-loop contributions are cancelled by the coupling constant renormalization in the one-loop contributions. This has the consequence that $\gamma_{\mathcal{O}}(\{g_{R_{jk}}\})$ has a finite limit as $\varepsilon \rightarrow 0$, that is B_2 vanishes. This is shown in appendix C.

Now we set

$$g_{R_{jk}} = g_R^* = \frac{\varepsilon}{b_d} \quad \text{for all } (j, k) \tag{4.31}$$

following (4.19). Then we have

$$\gamma_{\mathcal{O}}^* = B_1^* \varepsilon + B_3^* \varepsilon^2 + \mathcal{O}(\varepsilon^2) \tag{4.32}$$

with

$$\begin{aligned}
 B_1^* & = -\frac{1}{2} \sum_{1 \leq j_1 < j_2 \leq p} n_{j_1} n_{j_2} \\
 B_3^* & = -\frac{1}{2} \sum_{1 \leq j_1 < j_2 < j_3 \leq p} n_{j_1} n_{j_2} n_{j_3} \left\{ \ln R(D_{j_1}, D_{j_2}, D_{j_3}) + \ln R(D_{j_2}, D_{j_3}, D_{j_1}) \right. \\
 & \quad \left. + \ln R(D_{j_3}, D_{j_1}, D_{j_2}) \right\} - \frac{1}{4} \sum_{1 \leq j_1 < j_2 \leq p} n_{j_1} n_{j_2} \left\{ (n_{j_1} - 1) \ln R(D_{j_1}, D_{j_2}, D_{j_1}) \right. \\
 & \quad \left. + (n_{j_2} - 1) \ln R(D_{j_2}, D_{j_1}, D_{j_2}) \right\} \\
 & = -\frac{1}{2} \sum_{1 \leq j_1 < j_2 < j_3 \leq p} n_{j_1} n_{j_2} n_{j_3} \ln \left(\frac{(D_{j_1} D_{j_2} + D_{j_1} D_{j_3} + D_{j_2} D_{j_3})^3}{(D_{j_1} + D_{j_2})^2 (D_{j_2} + D_{j_3})^2 (D_{j_3} + D_{j_1})^2} \right) \\
 & \quad - \frac{1}{4} \sum_{1 \leq j_1 < j_2 \leq p} n_{j_1} n_{j_2} \left\{ n_{j_1} \ln R(D_{j_1}, D_{j_2}, D_{j_1}) + n_{j_2} \ln R(D_{j_2}, D_{j_1}, D_{j_2}) \right\} \\
 & \quad + \frac{1}{4} \sum_{1 \leq j_1 < j_2 \leq p} n_{j_1} n_{j_2} \ln \left(\frac{D_{j_1} D_{j_2} (D_{j_1} + 2D_{j_2}) (2D_{j_1} + D_{j_2})}{(D_{j_1} + D_{j_2})^4} \right).
 \end{aligned}$$

Through (4.17), this gives the result (2.3).

4.7. Two dimensions

When $d = 2$ the couplings $\{g_{Rjk}\}$ are marginally irrelevant, that is they flow logarithmically slowly towards zero. In that case, it is not sufficient to set them equal to their fixed-point values, but instead the full Callan–Symanzik equation (4.14) must be solved. Using the fact that $\overline{G}_{\mathcal{O}R}(t, \{g_{Rjk}\}, \sigma)$ depends on σ and t only through combination (σt) , this may be rewritten

$$\left(t \frac{\partial}{\partial t} - \gamma_{\mathcal{O}}(\{g_{Rjk}\}) + \sum_{1 \leq j < k \leq p} \beta_{jk}(g_{Rjk}) \frac{\partial}{\partial g_{Rjk}} \right) \overline{G}_{\mathcal{O}R}(t, \{g_{Rjk}\}, \sigma) = 0. \quad (4.33)$$

The solution by the method of characteristics is standard. Define running couplings $\{\tilde{g}_{jk}(u)\}$ by

$$u \frac{d}{du} \tilde{g}_{jk}(u) = -\beta_{jk}(\tilde{g}_{jk}(u)) \quad (4.34)$$

with initial conditions $\tilde{g}_{jk}(1) = g_{Rjk}$. Then

$$\overline{G}_{\mathcal{O}R}(t, \{g_{Rjk}\}, \sigma) = \exp \left\{ \int_1^{\sigma t} \gamma_{\mathcal{O}}(\{\tilde{g}_{Rjk}\})(du/u) \right\} \overline{G}_{\mathcal{O}R}(\sigma^{-1}, \{\tilde{g}_{jk}(\sigma t)\}, \sigma). \quad (4.35)$$

In our case,

$$\tilde{g}_{jk}(u) = \frac{1}{\frac{1}{2}b_2 \ln u + g_{Rjk}^{-1}} = \frac{2}{b_2 \ln u} + \mathcal{O}((\ln u)^{-2}) \quad (4.36)$$

so that, using (4.22) the exponent in (4.35) is

$$\ln \ln(\sigma t) \sum_{1 \leq j_1 < j_2 \leq p} n_{j_1} n_{j_2} + \mathcal{O}((\ln(\sigma t))^{-1}). \quad (4.37)$$

Exponentiating this yields the result quoted in (2.17). The dependence of the last factor in (4.35) on $\{\tilde{g}_{jk}(\sigma t)\}$ also generates corrections which are down by $\mathcal{O}((\ln(\sigma t))^{-1})$. All the non-universal behaviour resides in these, and higher order, corrections. Note the absence of corrections $\mathcal{O}(\ln \ln t / \ln t)$, which may be traced to the lack of higher-order terms in the β -functions (4.18).

5. Discussion

We have presented a generalization of the vicious walker problem in which walkers from different families annihilate on meeting, but walkers from the same family ignore each other. We have studied the problem in a field-theoretic renormalization group framework, suitable for understanding universal quantities such as critical exponents. We have focussed on the probability that all walkers have survived up to time t , and we have showed that, in dimension $d < 2$, this decays as a power, $t^{-\alpha(\{n_j\})}$, where n_j is the number of walkers in the j th family. While this result is true to all orders in $\varepsilon \equiv 2 - d$, the actual values of the exponents can, by this method, be evaluated only as a power series in ε , which we have carried out to second order. The coefficient of the $\mathcal{O}(\varepsilon^2)$ term depends on the ratios of the diffusivities of each family as well as the $\{n_j\}$. The lack of dependence of the $\mathcal{O}(\varepsilon)$ term on this ratio may be traced mathematically to the fact that the same bubble diagram enters into the coupling constant renormalization (figure 5) and the renormalization of the composite operator \mathcal{O}^* (figure 3).

For the same reason, the exponent for $N = 2$ walkers does not depend on the ratio of their diffusivities, because in this case the one-loop result is correct to all orders (as long as we do not expand the coefficients in powers of ε). The same would be true, to first order for

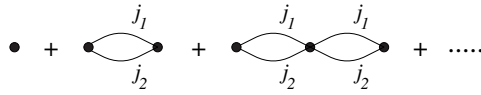


Figure 5. Diagrams renormalising $\lambda_{j_1 j_2}$.

$N > k$, and to all orders for $N = k$, if the two-body interactions we consider were generalized to k -body interactions with $k > 2$, although in this case the upper critical dimension would be reduced to $d_c(k) = 2/(k - 1)$.

The value of $\alpha(\{n_j\})$ was shown to be related to the anomalous dimension of a certain composite operator $\mathcal{O}_{\{n_j\}}^* = \prod_j \phi_j^{*n_j}$. This structure means that our results can be straightforwardly extended to other physical observables. For example, the *reunion* probability $R(t)$ that all the walkers have survived up to time t and are all located within a distance $\mathcal{O}(b) \ll t^{1/2}$ of each other (where b is, for example, the lattice spacing), is related to a correlation function

$$R(t) \sim \int \langle \mathcal{O}(t, \mathbf{r}) \mathcal{O}^*(0, \mathbf{0}) \rangle d^d r \tag{5.1}$$

where $\mathcal{O} \equiv \prod_j \phi_j^{n_j}$. Since the theory is symmetric under $(t \rightarrow -t, \phi_j \leftrightarrow \phi_j^*)$, \mathcal{O} has the same anomalous dimension as \mathcal{O}^* . Hence

$$R(t) \sim \text{const } t^{-(N-1)(d/2) - 2\alpha(\{n_j\})} \tag{5.2}$$

where the first term in the exponent comes from simple power counting, and the factor of 2 in the second reflects the important fact that the anomalous scaling of composite operators at different times (and points) is multiplicative.

The fact that $\mathcal{O}_{\{n_j\}}^*$ is symmetric under permutations of the families $j = 1, \dots, p$ has the important consequence that the exponents $\alpha(\{n_j\})$ are also symmetric. The form of the ε -expansion implies that this is true to all orders in ε . It may be traced to the operator product expansion (3.18): the next-to-leading terms on the right-hand side, which do not vanish on integration over the spatial coordinates must contain at least two derivatives, for example,

$$(\nabla\phi_1 \cdot \nabla\phi_2)\phi_1^{n_1-1}\phi_2^{n_2-1} \prod_{3 \leq j \leq p} \phi_j^{n_j}. \tag{5.3}$$

Power counting shows that, at $d = 2$, the contribution of such terms is at least $\mathcal{O}(t^{-1})$ down on the leading term, and therefore, for sufficiently small ε , they yield only corrections to the leading behaviour which we have computed. However, since each term in (3.18) is renormalized separately, each gives rise to an independent scaling exponent. It is a very interesting question whether the first term in the OPE is dominant all the way down to $d = 1$. Such a result would imply that the asymptotic exponents (but not necessarily the prefactors) for the cases $\{n_1, n_2, n_3\} = \{2, 1, 1\}$ and $\{1, 2, 1\}$, for example, are equal. Yet, in one dimension, the two problems are certainly not isomorphic, because the ordering of the families along the real line makes a difference. A very similar situation occurs in the problem of intersections of families of Brownian paths in two dimensions: in that case the exponents have been computed exactly [12, 13] and are known to be symmetric, despite the fact that the ordering of the families around the annulus is *a priori* relevant. In this example, this symmetry is also suggested by the ε -expansion below the upper critical dimension, in this case $d_c = 4$ [11]. However, a recent result of Bray and Blythe [18] suggests that the situation in $d = 1$ is not so straightforward for our problem. In their equation (4) they report a result for the survival exponent for a single lamb, with diffusion constant D' , with N_L lions to the left and N_R lions to the right.

The diffusion constant of the lions is D , and their result is an expansion to first order in D'/D . It is reported to depend on the asymmetry $N_L - N_R$ as well as the sum $N_L + N_R$, while our analysis shows that it should depend only on the sum, to all orders in the ε -expansion.

There are at least two possible resolutions:

1. the qualitative conclusions of the ε -expansion break down in $d = 1$, either because the non-leading terms in the OPE dominate, or through some more systemic failure;
2. for $d = 1$ there is a qualitative difference between the case when walkers of different families strictly cannot pass each other, and that, more appropriate to the field theory approach, when the annihilation rate is finite and, therefore, the order along the real line is not preserved.

In any case, it may be shown, by generalizing the arguments of Krapivsky and Redner [7–9], that for infinite annihilation rate in $d = 1$ the exponents $\alpha(\{n_j\})$ (and indeed all the non-leading exponents), are simply related to the eigenvalues of the Dirichlet problem in a certain compact region consisting of a spherical hyperpolygon on the sphere S_{N-2} . Thus, if the postulated symmetry were to hold, it would imply that the Laplacians in different regions which are related by permutations of the $\{n_j\}$ are isospectral. We plan to address this question in a future publication.

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Appendix A. Coupling constant renormalization

The diagrams contributing to $\Gamma_{j_1 j_2}^{(2,2)}$ are shown in figure 5. In (s, \mathbf{q}) space they give a geometric sum. We have therefore the exact result

$$\Gamma_{j_1 j_2}^{(2,2)}((s'_{j_1}, \mathbf{q}'_{j_1}), (s'_{j_2}, \mathbf{q}'_{j_2}); (s_{j_1}, \mathbf{q}_{j_1}), (s_{j_2}, \mathbf{q}_{j_2})) = \frac{\lambda_{j_1 j_2}}{1 + \lambda_{j_1 j_2} \int \frac{d^d k}{(2\pi)^d} \frac{1}{s + D_{j_1} \mathbf{k}^2 + D_{j_2} (\mathbf{q} - \mathbf{k})^2}} \quad (\text{A.1})$$

where $s \equiv s'_{j_1} + s'_{j_2} = s_{j_1} + s_{j_2}$ and $\mathbf{q} \equiv \mathbf{q}'_{j_1} + \mathbf{q}'_{j_2} = \mathbf{q}_{j_1} + \mathbf{q}_{j_2}$. The renormalized coupling is the value of this at $s'_{j_1} = s'_{j_2} = s_{j_1} = s_{j_2} = \sigma$ and $\mathbf{q}'_{j_1} = \mathbf{q}'_{j_2} = \mathbf{q}_{j_1} = \mathbf{q}_{j_2} = 0$. Thus

$$\lambda_{Rj_1 j_2} = \frac{\lambda_{j_1 j_2}}{1 + \lambda_{j_1 j_2} \hat{I}_1(\sigma; D_{j_1}, D_{j_2})} \quad (\text{A.2})$$

where

$$\hat{I}_1(\sigma; D_{j_1}, D_{j_2}) = \int \frac{d^d k}{(2\pi)^d} \int_0^\infty d\alpha e^{-\alpha(2\sigma + (D_{j_1} + D_{j_2})\mathbf{k}^2)} \quad (\text{A.3})$$

$$= \frac{1}{(2\pi)^d} \left(\frac{\pi}{D_{j_1} + D_{j_2}} \right)^{d/2} \int_0^\infty d\alpha \alpha^{-d/2} e^{-\alpha(2\sigma)} \quad (\text{A.4})$$

$$= \frac{\Gamma(1 - d/2)}{2^{3d/2} \pi^{d/2}} \left(\frac{2}{D_{j_1} + D_{j_2}} \right)^{d/2} (2\sigma)^{-\varepsilon/2}. \quad (\text{A.5})$$

The dimensionless coupling (4.4) is then given as

$$g_{R_{j_1 j_2}} = \frac{\lambda_{j_1 j_2} \left(\frac{2}{D_{j_1} + D_{j_2}} \right)^{d/2} (2\sigma)^{-\varepsilon/2}}{1 + \frac{b_d}{\varepsilon} \lambda_{j_1 j_2} \left(\frac{2}{D_{j_1} + D_{j_2}} \right)^{d/2} (2\sigma)^{-\varepsilon/2}}. \quad (\text{A.6})$$

Thus

$$\lambda_{j_1 j_2} \left(\frac{D_{j_1} + D_{j_2}}{2} \right)^{-d/2} (2\sigma)^{-\varepsilon/2} = \frac{g_{R_{j_1 j_2}}}{1 - g_{R_{j_1 j_2}} \frac{b_d}{\varepsilon}}. \quad (\text{A.7})$$

Differentiating this equation with respect to σ at fixed $(\lambda_{j_1 j_2}, D_{j_1}, D_{j_2})$ and using the definition of the β -function (4.12)

$$(-\varepsilon/2) \frac{g_{R_{j_1 j_2}}}{1 - g_{R_{j_1 j_2}} \frac{b_d}{\varepsilon}} = \left(\frac{1}{1 - g_{R_{j_1 j_2}} \frac{b_d}{\varepsilon}} \right)^2 \beta_{j_1 j_2}(g_{R_{j_1 j_2}}). \quad (\text{A.8})$$

That is

$$\beta_{j_1 j_2}(g_{R_{j_1 j_2}}) = -\frac{1}{2} (\varepsilon g_{R_{j_1 j_2}} - b_d g_{R_{j_1 j_2}}^2). \quad (\text{A.9})$$

Appendix B. The integral \hat{I}_2

At the normalization point, the two-loop diagram figure 4(c) leads to the integral

$$\begin{aligned} \hat{I}_2(\sigma; D_1, D_2, D_3) &= \int \frac{d^d q}{(2\pi)^d} \int \frac{d^d k}{(2\pi)^d} \frac{1}{(2\sigma + (D_2 + D_3)\mathbf{k}^2)(3\sigma + D_1\mathbf{q}^2 + D_2(\mathbf{k} + \mathbf{q})^2 + D_3\mathbf{k}^2)}. \end{aligned} \quad (\text{B.1})$$

As usual, the denominators may be combined using a Feynman parameter integration over x :

$$\hat{I}_2(\sigma; D_1, D_2, D_3) = \int_0^1 dx \int \frac{d^d q}{(2\pi)^d} \int \frac{d^d k}{(2\pi)^d} \frac{1}{((D_2 + D_3)\mathbf{k}^2 + x(D_1 + D_2)\mathbf{q}^2 + 2x D_2 \mathbf{k} \cdot \mathbf{q})^2}. \quad (\text{B.2})$$

The wave number integrals are now standard, and yield

$$\hat{I}_2 = \frac{\pi^d}{(2\pi)^d} \Gamma(2 - d) (D_1 + D_2)^{-d/2} (D_2 + D_3)^{-d/2} \sigma^{d-2} \times J \quad (\text{B.3})$$

where

$$J = \int_0^1 dx x^{-d/2} (2+x)^{d-2} \left(1 - \frac{x D_2^2}{(D_1 + D_2)(D_2 + D_3)} \right)^{-d/2}. \quad (\text{B.4})$$

This has a simple pole at $d = 2$, arising from the end point at $x = 0$. However, we also have to extract the finite part. This may be done by writing $J = J_1 + J_2$ where

$$J_1 = \int_0^1 dx x^{-d/2} 2^{d-2} = \frac{2}{\varepsilon} 2^{d-2} \quad (\text{B.5})$$

$$J_2 = \int_0^1 dx x^{-d/2} \left[(2+x)^{d-2} \left(1 - \frac{x D_2^2}{(D_1 + D_2)(D_2 + D_3)} \right)^{-d/2} - 2^{d-2} \right]. \quad (\text{B.6})$$

The second integral is finite at $d = 2$:

$$\begin{aligned} J_2 &= \int_0^1 \frac{dx}{x} \left[\left(1 - \frac{x D_2^2}{(D_1 + D_2)(D_2 + D_3)} \right)^{-1} - 1 \right] + \mathcal{O}(\varepsilon) \\ &= -\ln R(D_1, D_2, D_3) + \mathcal{O}(\varepsilon) \end{aligned} \quad (\text{B.7})$$

where

$$R(D_1, D_2, D_3) \equiv \frac{D_1 D_2 + D_2 D_3 + D_1 D_3}{(D_1 + D_2)(D_2 + D_3)}. \quad (\text{B.8})$$

Recalling the definition (4.3) of b_d , we therefore find, after some algebra,

$$\frac{\hat{I}_2}{b_d^2} = \left(\frac{2}{D_1 + D_2} \right)^{d/2} \left(\frac{2}{D_2 + D_3} \right)^{d/2} (2\sigma)^{-\varepsilon} \left[\frac{1}{2\varepsilon^2} - \frac{1}{4\varepsilon} \ln R + \mathcal{O}(1) \right] \quad (\text{B.9})$$

which leads directly to (4.28). It should be remarked that a crucial simplification in this calculation arises because

$$\frac{\Gamma(2-d)}{\Gamma(1-d/2)^2} = \frac{\varepsilon}{4} (1 + \mathcal{O}(\varepsilon^2)). \quad (\text{B.10})$$

Appendix C. Verification that $B_2 = 0$

By (4.30),

$$\begin{aligned} B_2 &= \frac{1}{2} \sum_{1 \leq j_1 < j_2 \leq p} n_{j_1} n_{j_2} (g_{R_{j_1 j_2}})^2 \\ &\quad + \sum_{1 \leq j_1 < j_2 < j_3 < j_4 \leq p} n_{j_1} n_{j_2} n_{j_3} n_{j_4} (g_{R_{j_1 j_2} R_{j_3 j_4}} + g_{R_{j_1 j_3} R_{j_2 j_4}} + g_{R_{j_1 j_4} R_{j_2 j_3}}) \\ &\quad + \sum_{1 \leq j_1 < j_2 < j_3 \leq p} n_{j_1} n_{j_2} n_{j_3} (n_{j_1} g_{R_{j_1 j_2} R_{j_1 j_3}} + n_{j_2} g_{R_{j_1 j_2} R_{j_2 j_3}} + n_{j_3} g_{R_{j_1 j_3} R_{j_2 j_3}}) \\ &\quad - \sum_{1 \leq j_1 < j_2 < j_3 \leq p} n_{j_1} n_{j_2} n_{j_3} (g_{R_{j_1 j_2} R_{j_1 j_3}} + g_{R_{j_1 j_2} R_{j_2 j_3}} + g_{R_{j_1 j_3} R_{j_2 j_3}}) \\ &\quad + \frac{1}{2} \sum_{1 \leq j_1 < j_2 \leq p} n_{j_1}^2 n_{j_2}^2 (g_{R_{j_1 j_2}})^2 - \frac{1}{2} \sum_{1 \leq j_1 < j_2 \leq p} n_{j_1} n_{j_2} (n_{j_1} + n_{j_2}) (g_{R_{j_1 j_2}})^2 \\ &\quad + \frac{1}{2} \sum_{1 \leq j_1 < j_2 \leq p} n_{j_1} n_{j_2} (g_{R_{j_1 j_2}})^2 \\ &\quad + \sum_{1 \leq j_1 < j_2 < j_3 \leq p} n_{j_1} n_{j_2} n_{j_3} (g_{R_{j_1 j_2} R_{j_2 j_3}} + g_{R_{j_2 j_3} R_{j_1 j_3}} + g_{R_{j_1 j_3} R_{j_2 j_3}}) \\ &\quad + \frac{1}{2} \sum_{1 \leq j_1 < j_2 \leq p} n_{j_1} n_{j_2} (n_{j_1} + n_{j_2}) (g_{R_{j_1 j_2}})^2 \\ &\quad - \sum_{1 \leq j_1 < j_2 \leq p} n_{j_1} n_{j_2} (g_{R_{j_1 j_2}})^2 - \frac{1}{2} \left(\sum_{1 \leq j_1 < j_2 \leq p} n_{j_1} n_{j_2} g_{R_{j_1 j_2}} \right)^2 \end{aligned}$$

$$\begin{aligned}
&= \sum_{1 \leq j_1 < j_2 < j_3 < j_4 \leq p} n_{j_1} n_{j_2} n_{j_3} n_{j_4} (g_{R_{j_1 j_2}} g_{R_{j_3 j_4}} + g_{R_{j_1 j_3}} g_{R_{j_2 j_4}} + g_{R_{j_1 j_4}} g_{R_{j_2 j_3}}) \\
&+ \sum_{1 \leq j_1 < j_2 < j_3 \leq p} n_{j_1} n_{j_2} n_{j_3} (n_{j_1} g_{R_{j_1 j_2}} g_{R_{j_1 j_3}} + n_{j_2} g_{R_{j_1 j_2}} g_{R_{j_2 j_3}} + n_{j_3} g_{R_{j_1 j_3}} g_{R_{j_2 j_3}}) \\
&+ \frac{1}{2} \sum_{1 \leq j_1 < j_2 \leq p} n_{j_1}^2 n_{j_2}^2 (g_{R_{j_1 j_2}})^2 - \frac{1}{2} \left(\sum_{1 \leq j_1 < j_2 \leq p} n_{j_1} n_{j_2} g_{R_{j_1 j_2}} \right)^2 \\
&= 0.
\end{aligned} \tag{C.1}$$

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